On Credit Risk Modeling and Management

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A first touch on Credit Risk

Definition of Credit Risk

Definition (Credit Risk)

Credit Risk is the risk of *economic loss* due to the failure of a counterparty to fulfill its contractual obligations (i.e. due to *default*)

- Thus we are interested in
  - how likely it is that we will not receive the promised payments from our counterparty (i.e. *Probability of default*)
  - how much of these promised payments we can get back in case the counterparty defaults (i.e. *Recovery rate* of our exposure)
- Eventually we are interested in the *distribution* of potential losses due to the uncertainty surrounding the credibility of our counterparties

- For modeling reasons we would need good realistic working definitions for "default" and for "economic loss".
- Then, based on our definition of "default" we could proceed to estimating the potential "economic loss"
- The concept of default may refer to the missing of a payment in due time, the deterioration of credibility of the counterparty or whatever one may want to consider as a credit event.
- Default does not mean necessarily bankruptcy.
- Furthermore, default does not necessarily imply losses!!
- There is no consensus yet for the "best" definition of default!

On the definition of Default

- Bank for International Settlements (BIS): "the obligor is unlikely to pay its credit obligations or the obligor is past due more than 90 days on any material credit obligation"
  - This is clearly an unclear definition
  - Furthermore, in retail banking especially, it is very usual that obligors repay their debt in full *after* the passage of these 90 days. Very often this has to do with negligence rather than inability to repay. In these cases default does not lead to credit losses (rather the opposite!)
- Default defined as bankruptcy
  - e.g. USA Chapter 11 Bankruptcy or other national laws for bankruptcy
  - However a firm may default on payments and still not declare official bankruptcy.

Drivers of Credit Risk

Definition (Drivers of Credit Risk)

The distribution of loss due to credit risk is driven by the following variables

- Default
- EAD: Exposure at Default
- RR: Recovery rate (the fractional amount of the exposure that will be recovered in the case of default)
  Instead of RR one may equivalently consider the "Loss Given Default (LGD) = the fractional loss of EAD".

$$LGD = 1 - RR$$

Each of these factors of credit risk need to be modelled and then combined to translate into a loss variable

Remarks

- In the case that the exposure is to a derivative instrument, this may have positive or negative value at the time of default. In this case $EAD = \max(Value \;\; of \;\; derivative, 0)$
  This means that if *we* owe money to a counterparty that goes bankrupt, we do not get rid of our obligations to them.
- A long credit position has by its nature an embedded short option position.
  This option has been sold by the lender and bought by the borrower and it reflects the right of the borrower to default.
  In this sense the required yield of a risky bond can be decomposed as the yield of a risk free bond plus the option premium.

$$y_{risky} = y_{risk-free} + spread$$

The basic model

The credit loss from an individual borrower i can be modeled as

$$CL_i = b_i EAD_i \underbrace{(1 - RR_i)}_{LGD_i}$$

where

$$b_i = \begin{cases} 1 & \text{with probability } p_i & \text{(Default)} \\ 0 & \text{with probability } 1 - p_i & \text{(No Default)} \end{cases}$$

$EAD_i =$ credit exposure at time of default

$RR_i =$ recovery rate

$LGD_i =$ loss given default

The basic model - Portfolio

- Let Portfolio of N loans to different obligors
  Then the Credit Loss of the portfolio is the random variable

$$CL = \sum_{i=1}^{N} CL_i = \sum_{i=1}^{N} b_i EAD_i \underbrace{(1 - RR_i)}_{LGD_i}$$

- If all independent, then

$$E(CL) = \sum_{i=1}^{N} E(b_i)E(EAD_i)E(LGD_i) = \sum_{i=1}^{N} p_i E(EAD_i)E(LGD_i)$$

A simple but instructive case (homogeneous portfolio with independent drivers)

- Let again Portfolio of N loans to different obligors
- b_i are i.i.d., LGD_i are i.i.d., $EAD_i = 1$
- Then

$$CL = \sum_{i=0}^{n} LGD_i$$

  where n is the number of defaults in N trials.
- Can we say anything about the Expected value and the standard deviation of the portfolio credit loss?

Expected credit loss:

$$E(CL) = E(E(CL|n)) = E(E(\sum_{i=0}^{n} LGD_i|n))$$

$$= E(nE(LGD)) = E(n)E(LGD) = NpE(LGD)$$

- Therefore E(CL) linear with respect to p

Variance of the portfolio credit loss:

$$Var(CL) = EVar(CL|n) + Var(E(CL|n))$$

$$= E(Var(\sum_{i=1}^{n} LGD_i|n)) + Var(E(\sum_{i=1}^{n} LGD_i|n))$$

$$= E(nVar(LGD)) + Var(nE(LGD))$$

$$= E(n)Var(LGD) + Var(n)E(LGD)^2$$

$$= NpVar(LGD) + Np(1-p)E(LGD)^2$$

- Therefore the standard deviation of the Credit Loss depends on both $Var(LGD)$ and $Var(b) = p(1-p)$ and is not linear on p.

- Suppose for an instance that N=1
- For small p the standard deviation is approximately proportional to $\sqrt{p}$ so in this case $stdev(CL)$ increases faster than $E(CL)$ with p.
- If we think of $stdev(CL)$ to represent a measure of unexpected losses, then we can say that as credit quality deteriorates (i.e. higher default rates) then unexpected losses increase much faster than the expected losses.
- Keep in mind that although the expected losses are already priced in the interest rate that a bank charges, the unexpected losses are covered by the bank's own equity capital.

If the total exposure of the portfolio is F and each of the N obligors has the same $EAD_i = \frac{F}{N}$ then the previous reasoning gives:

- $$CL = \frac{F}{N}\sum_{i=0}^{n} LGD_i$$

  where n is the number of defaults in N trials.

- $$E(CL) = pE(LGD)F$$

- $$Var(CL) = \frac{F^2}{N}\left(pVar(LGD) + p(1-p)E(LGD)^2\right)$$

- Notice that as N grows the variance tends to 0, i.e. the diversification effect decreases the uncertainty about the actual credit losses, which get close to their expected value.

Joint Defaults and the Diversification effect

Proposition

- *The expected credit loss depends on default probabilities but not on the correlation between the default variables*
- *However, the variance of credit losses (i.e. unexpected loss) depends significantly on the correlation between defaults.*

To see this consider a total exposure F to N borrowers and suppose zero recovery rate in case of default
Let

- w_i the percentage of F that has been lent to borrower i
- b_i the default indicator variable of borrower i.
  (It is Bernoulli with $E(b_i) = p_i$=probability of default of i, and $\sigma_i = \sqrt{p_i(1 - p_i)}$)
- (ρ_{ij}) the correlation matrix of b_i's
- Let $CL_i = b_i w_i F$ the loss variable of borrower i and

$$CL = \sum_{i=1}^{N} b_i w_i F$$

the loss variable of the portfolio

Then

- $$E(CL) = F \sum_{i=1}^{N} w_i E(b_i) = F \underbrace{\sum_{i=1}^{N} w_i p_i}_{\text{average default rate}}$$

  independent of default correlations
- In particular if $p_1 = ... = p_N = p$ (i.e. the borrowers belong to the same class), then

  $$E(CL) = Fp$$

  , linear in p.

- $$Var(CL) = Var(\sum_{i=1}^{N} b_i w_i F) =$$
$$F^2(\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij})$$
- If $w_1 = ... = w_N = 1/N$, (i.e. the portfolio does not exhibit concentrations) then the last expression becomes

$$Var(CL) = \frac{F^2}{N} \underbrace{\left(\frac{\sum \sigma_i^2}{N} \right)}_{\text{average variance}} + F^2 \frac{N-1}{N} \underbrace{\left(\frac{1}{N(N-1)} \sum \sum_{i \neq j} \sigma_{ij} \right)}_{\text{average covariancel}}$$

- If moreover all the borrowers have the same prob. of default p, then the last expression becomes

$$Var(CL) =$$

$$\frac{F^2}{N} \underbrace{\left(\frac{\sum \sigma_i^2}{N}\right)}_{\text{average variance}} + F^2 \frac{N-1}{N} \underbrace{p(1-p)}_{\text{variance of default}} \underbrace{\left(\frac{1}{N(N-1)} \sum \sum_{i \neq j} \rho_{ij}\right)}_{\text{average correlation}}$$

- Notice now that as N gets large, the variance of the portfolio loss tends to $p(1-p) * (\text{average correlation}) * F^2$.
- Therefore large portfolios with lower default correlations have lower uncertainty.

Example (Trivial Example for portfolio distribution)

Three companies A, B, C with fixed exposures at default, zero recovery rate and independent defaults

	EAD	PD
A	100	10%
B	200	5%
C	250	7%

Example (continued)

Then we can form the portfolio loss distribution

Defaults	Loss	Probability	Expected Loss	stdev
None	0	79,52%	0	
A	100	8,84%	8,83	
B	200	4,19%	8,37	
C	250	5,99%	14,97	
A,B	300	0,47%	1,39	
A,C	350	0,67%	2,33	
B,C	450	0,32%	1,42	
A,B,C	550	0,04%	0,19	
		100,00%	37,5	82,88

and from there we can read credit risk measures like VaR.

Example (The diversification effect)

Three companies A, B, C with fixed exposures at default, zero recovery rate and independent defaults

	EAD	PD
A	100	5%
B	100	5%
C	100	5%

The total exposure in this case is 300, the Expected loss is 15 and the standard deviation is 37,75

However if we had only one exposure of 300 with PD=5% then the Expected loss would be again 15 but the standard deviation would be 65,38 which is much higher and reflecting the effect of low correlations in the diversified case.

More on default correlation

- Example of positive correlation: a firm is creditor of another firm
  (A buys materials from B with credit. If A faces cashflow problems then B will face cashflow problems)
- Example of negative correlation: two firms are competitors
  (The increased sales of one affect negatively the increased sales of the other)
- Obvious drivers: state of the economy, industry specific factors

More on the importance of default correlation

- Two firms
- Default probabilities $p_1,\ p_2$
- Joint default probability $p_{1,2}$
- Conditional default probability $p_{1|2}$. Clearly , $P_{1|2} = \frac{p_{1,2}}{p_2}$
- Correlation of defaults $\rho 1, 2$

- $\rho 1, 2 = \frac{p_{1,2} - p_1 p_2}{\sqrt{p_1(1-p_1)p_2(1-p_2)}}$
- Assuming small equal probabilities of default we have:

$$p_{1,2} = p_1 p_2 + \rho_{1,2}\sqrt{p_1(1-p_1)p_2(1-p_2)} \sim p^2 + \rho_{1,2}p \sim \rho_{1,2}p$$

and

$$p_{1|2} = \frac{p_{1,2}}{p_2} = p_1 + \frac{\rho_{1,2}}{p_2}\sqrt{p_1(1-p_1)p_2(1-p_2)} \sim \rho_{1,2}$$

Sources of Information and Passage to Modern Credit Risk Modelling

Sources of information

Recall the basic drivers of (individual) credit risk:

- The default process and the probability of default
- The recovery rate (equivalently the loss given default)

Where from can we get information about them?

- From the counterparty itself (form our own opinion)
- From the market (ask others opinion)

From the counterparty itself

- Look at its fundamentals and form our own opinion from first hand (get info from balance sheet, business plans, cashflows, financial ratios etc.).
- Based on this opinion we may be able to express our own well informed (?) views about the "right" price (absolute or relative) of the various tradables that are related to the firm (e.g. bonds, equity, CDS's etc).
- In particular we may express an opinion that is different than the market's opinion and then trade on this difference.

From the market

- Accept that the market knows better and look at ”what the market has to say” about the firm.
- Prices of bonds issued by the firm or prices of other tradable instruments that are related to our counterparty may be appropriate for revealing such information.
- Based on this info, price new issues or more exotic instruments on the firm, in accordance to market’s beliefs.

The two basic modern approaches to Credit Risk Modelling

In correspondence to the previous two sources of information there are two main approaches to credit risk modelling

- The Structural approach
- The Reduced form approach

Both of these approaches rely on the options pricing theory of Black and Scholes (1973).

Structural models

- Basic Idea: Merton (1974)
- Look directly at the financial structure of the firm
- Provide explicit relationship between risk of default and capital structure (economic insight)
- The default is the consequence of some company event, e.g. insufficient value of assets to cover a debt payment
- Attempt to say at which price the corporate bonds "should" trade
- Based on the internal capital structure of the firm (therefore require balance sheet information)
- Establish link between equity and debt market
- However, hard to calibrate (balance sheet data frequency), lack of flexibility to fit well observed term structures, not easy extension for derivatives pricing.

Reduced form models

- Basic Idea: Jarrow-Turnbull (1995)
- Direct modelling of default as exogenous event driven by a stochastic process (e.g. Poisson or Cox process)
- The probability of default extracted from market prices together with a pricing model
- Easy calibration to market prices
- Can be easily extended to price more exotic derivatives, thus more easily used for credit derivatives pricing
- However no explanation of the mechanics of default, together with exogenously given recovery rate (i.e. no economic insight)
- In particular they do not allow for the analysis of strategic decisions to default or renegotiations

Merton's Model (1974)

Merton's Model

Default occurs when the value of the assets of the company are not sufficient to service its debt

- Black-Scholes Economy setup

  Trading does not affect prices (i.e. agents are price takers)
  No transaction costs
  No indivisibility of assets
  Unlimited short selling
  Unlimited borrowing or lending at the risk free rate r

- Future uncertainty is represented via a filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathcal{P})$ with $(\mathcal{F}_t)$ generated by a standard Brownian motion W_t.

- The Bank account for borrowing or lending at the risk free rate r: $\beta_0 = 1, \beta_t = exp(r \cdot t)$

- The Firm's structural variables

  (V_t): Assets process
  (B_t): Liabilities process
  (E_t): Equity process
  related by the Accounting Identity $V_t = B_t + E_t$

- The dynamics of the assets

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t$$

which is equivalent by Ito's lemma to

$$V_t = V_0 \exp((\mu_V - \frac{\sigma_V^2}{2})t + \sigma_V W_t)$$

from which follows that

$$\ln(V_t) \sim N(\ln(V_0) + (\mu_V - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$$

The probability of default under the physical measure

Assume that default occurs at time T if $V_T < D$ for some threshold D
Then the last relation readily implies that

$$P.D. = \mathcal{P}(\ln(V_T) < \ln(D)) = \Phi(\frac{\ln(D) - \ln(V_0) - (\mu_V - \frac{\sigma_V^2}{2})T}{\sigma_V\sqrt{T}})$$

where Φ is the standard normal cumulative distribution.

- The Liabilities:
  Only a zero coupon bond with face value D maturing at time T
- What happens at maturity?
  (i) if $V_T \geq D \Rightarrow B_T = D = D - 0, \quad E_T = V_T - D$
  (ii) if $V_T < D \Rightarrow B_T = V_T, \quad E_T = D - (D - V_T)$
- Equivalently
$$B_T = D - \max(D - V_T,\ 0)$$
  and
$$E_T = \max(V_T - D,\ 0)$$
- Therefore
  Bondholders position=long a risk free asset and short a put option on firm's assets with strike the face value of debt
  Shareholders position=long call option on firm's assets with strike the face value of debt

Remark

- Put-Call parity allows to see the bonholders and shareholders positions in another way as well
- The bonholders position at maturity is $D - P$ but this is equal to $V - C$
- This can be interpreted as if the bondholders "own" the firm's assets but they have also sold to the shareholders the right to obtain the assets at time T by paying a price equal to the face value of the bond. In this sense the call option that the shareholders hold has been sold to them by the bondholders.

Conflict of interest between bondholders and shareholders

- Volatility of assets can be thought as expressing the riskiness of the firm's projects
- Increasing volatility increases the value of both the call and the put.
- The bondholders are short a put, thus they prefer lower volatility (i.e. less risky projects)
- The shareholders are long a call. thus they prefer higher volatility (i.e. more risky projects)

Valuation of Equity and Debt prior to Maturity

Under risk neutrality μ_V is being replaced by r.
In other words, there exist an equivalent martingale measure $\mathcal{Q}$ under which $V_t = V_0 \exp((r - \frac{\sigma_V^2}{2})t + \sigma_V W_t^Q)$. Then Black-Scholes imply:

$$E_t = C^{BS}(V_t, D, T, \sigma_V, r) \Longrightarrow$$

$$E_t = V_t \cdot \Phi(d_1) - D\exp(-r(T-t))\Phi(d_2)$$

where

$$d_1 = \frac{\ln(\frac{V_t}{D}) + r(T-t) + \frac{\sigma_V^2}{2}(T-t)}{\sigma_V\sqrt{T-t}} = \frac{\ln(\frac{V_t}{D\exp(-r(T-t))})}{\sigma_V\sqrt{T-t}} + \frac{\sigma_V}{2}\sqrt{(T-t}$$

$$d_2 = d_1 - \sigma_V\sqrt{T-t} = \frac{\ln(\frac{V_t}{D\exp(-r(T-t))})}{\sigma_V\sqrt{T-t}} - \frac{\sigma_V}{2}\sqrt{(T-t)}$$

- Similarly

$$B_t = D\exp(-r(T-t)) - P^{BS}(V_t, D, \sigma_V, r, T-t) \Longrightarrow$$

$$B_t = \exp(-r(T-t))D\Phi(d_2) + V_t\Phi(-d_1)$$

- Notice that $\Phi(d_2) = \mathcal{Q}(V_T \geq D)$ and therefore

$$1 - \Phi(d_2) = \Phi(-d_2)$$

is the probability of default under the risk neutral measure

The promised yield and the credit spread

At time t the holder of the bond is *promised* a continuously compounded annual return $y(t, T)$ for holding the bond until T.

$$B_t \exp(y(t, T)(T - t)) = D$$

So, *if* promises are kept (i.e. no default) the promised yield is:

$$y(t, T) = \frac{1}{T - t} \ln(\frac{D}{B_t})$$

The credit spread or yield spread is

$$s(t, T) = y(t, T) - r(t, T)$$

where $r(t, T) = r$ is the risk free yield for the period $[t, T]$

Credit Spread

Proposition

The credit spread depends only on the leverage of the firm and the volatility of assets

Indeed

$$
\begin{aligned}
s(t,T) &= \frac{1}{T-t}\ln(\frac{D}{B_t}) - \frac{1}{T-t}\ln(e^{r(T-t)}) \\
&= \frac{1}{T-t}\ln(\frac{D\exp(-r(T-t)}{B_t}) \\
&= \frac{-1}{T-t}\ln(\frac{B_t}{D\exp(-r(T-t)}) \\
&= \frac{-1}{T-t}\ln\left(\frac{\exp(-r(T-t))D\Phi(d_2) + V_t\Phi(-d_1)}{D\exp(-r(T-t)}\right)
\end{aligned}
$$

$$s(t,T) = \frac{-1}{T-t}\ln\left(\Phi(d_2) + \Phi(-d_1)\frac{V_t}{D\exp(-r(T-t))}\right)$$

The fraction $\frac{V_t}{D\exp(-r(T-t))}$ expresses the leverage of the firm while

$$d_1 = \frac{\ln(\frac{V_t}{De^{-r(T-t)}})}{\sigma_V\sqrt{T-t}} + \frac{\sigma_V\sqrt{T-t}}{2}$$

$$d_2 = \frac{\ln(\frac{V_t}{De^{-r(T-t)}})}{\sigma_V\sqrt{T-t}} - \frac{\sigma_V\sqrt{T-t}}{2}$$

are functions only of leverage and assets volatility as well.

The credit loss and the recovery rate

At time T the loss due to credit risk is

$$D - B_T$$

which is the time T value of the put option.
Therefore under risk neutrality the expected loss at time T is $e^{r(T-t)}$ times the time t value of the put option, i.e.

$$EL_T = \underbrace{\Phi(-d_2)}_{P.D.} \underbrace{\left[D - V_t e^{r(T-t)} \frac{\Phi(-d_1)}{\Phi(-d_2)} \right]}_{EAD \cdot (1 - RR)}$$

But $EAD = D$ and thus the previous relation

$$EAD \cdot (1 - RR) = D - V_t e^{r(T-t)} \frac{\Phi(-d_1)}{\Phi(-d_2)}$$

implies

$$RR = \frac{V_t \Phi(-d_1)}{De^{-r(T-t)} \Phi(-d_2)}$$

- The important thing however here is that the recovery rate is endogenously built in the model.

Seniority issues (subordinated debt)

Suppose we have 2 different priorities on the servicing of the debt, senior debt of face value D_S and junior debt of face value D_J

Example

Seniority	Payoff		
	$V_T < D_S$	$D_s \leq V_T < D_S + D_J$	$D_s + D_J < V_T$
Senior	V_T	D_S	D_S
Junior	0	$V_T - D_s$	D_J
Equity	0	0	$V_T - D_S - D_J$

Therefore:

Seniority	Payoff
Senior	$V_T - [V_T - D_S]^+$
Junior	$[V_T - D_S]^+ - [V_T - (D_S + D_J)]^+$
Equity	$[V_T - (D_S + D_J)]^+$

Implementation Problems

- Assets Value V_t and volatility of assets σ_V are unobservable
- The debt structure is too simple

Infering Assets value and volatility

In practice we observe E_t and σ_E.
From this we want to infer values for V_t and σ_V so that

$$E_t = V_t \Phi(d_1) - De^{-r(T-t)}\Phi(d_2)$$

This is a function of our two unknowns V_t and σ_V. So we need one more equation. Notice that since $\Phi(d_1)$ is the hedge ratio we have

$$dE_t = \Phi(d_1) dV_t \Rightarrow \ldots + \sigma_E E_t dW_t = \ldots + \Phi(d_1)\sigma_V V_t dW_t$$

which implies

$$\sigma_E E_t = \Phi(d_1)\sigma_V V_t$$

which is the second equation.

KMV model

- Variant of Merton's approach to track changes in credit risk for publicly traded firms
- KMV=Kealhofer, McQuown, Vasicek during the 90's (bought in 2002 by Moodys)
- References Crosbie and Bohn (2002), Crouhy,Galai and Mark (2000)

Quick description of KMV

1. Identify the default threshold to be used. KMV takes $\tilde{D}=$ FV of short term (<1 year) liabilities $+1/2$ FV of longterm liabilities.
2. Use $\tilde{D}, E_t, \sigma_E$ to identify V_t, σ_V
3. Introduce the concept of Distance to Default as $DD = \frac{V_0 - \tilde{D}}{\sigma_V V_0}$ which counts in standard deviations the distance of the firm's value from the default threshold $\tilde{D}$. This is done so that different firms can be compared.
4. Then make the assumption that equal DD 's correspond to equal EDF's (expected default frequencies). This correspondence is calculated by the use of their database where they can see the percentage of firms with a given DD that defaulted within a year.

Remark

Remark that in Merton's model the probability of default with one year horizon is given by

$$\Phi\left(\frac{\ln(\frac{D}{V_0}) - \mu_V + \sigma_V^2/2}{\sigma_V}\right)$$

Notice that $-DD = \frac{\tilde{D}-V_0}{\sigma_V V_0}$ *is an approximation of the argument of* Φ *since* $\ln(D/V_0) \sim (D - V_0)/V_0$ *and* μ_V, σ_V^2 *are small.*
Therefore one is tempted to write $PD = 1 - \Phi(DD)$
or interpret $\Phi(DD)$ *as the survival probability.*

More complicated liabilities (coupons)

- Suppose now that the bond has to make n payments $D_1, \ldots, D_n$ at times $t_1, \ldots t_n$. When we are at time $t_{n-1} < t \leq t_n$ we have a zero coupon bond which can be priced ala Merton as $V_t - C_t(V_t, D_n, t_n - t)$.
- At time t_{n-1} things are more complicated since the shareholders have to decide whether it is in their interest to default or not. Distinguish two cases
  - Asset sales are not allowed
  - Asset sales are allowed

Asset sales not allowed

Here the payment D_{n-1} will be paid out of the pockets of the shareholders
If they pay the amount D_{n-1} they will have in their hands a call option on assets value, with strike D_n maturing at time t_n. Let $C_{t_{n-1}}(V_t,\ D_n,\ t_n - t_{n-1})$ the value of this option at time t_{n-1}. If $D_{n-1} \leq C_{t_{n-1}}(V_t,\ D_n,\ t_n - t_{n-1})$ they will buy this option by paying D_{n-1}, otherwise they will default.

In other words their position at time t_{n-1}, before the payment of the amount D_{n-1}, is a long position on a call option that allows them to buy another call option. The value of their position is $\max(C_{t_{n-1}}(V_t,\ D_n,\ t_n - t_{n-1}) - D_{n-1}, 0)$. Similarly for previous times. Once the value of equity is known we can conclude the value of the bond by using the accounting identity.

Another way to look at it is as follows: At time t_{n-1} the shareholders find the value $\bar{V_{t_{n-1}}}$ of the firm's assets that solves the equation $D_{n-1} = C_{t_{n-1}}(V_t,\ D_n,\ t_n - t_{n-1})$. Then, if $V_{t_{n-1}} < \bar{V_{t_{n-1}}}$ they will default, otherwise they will not default.

The implementation of this model on a binomial tree is easy.

Assets sales allowed

Here the payment D_{n-1} will be paid by selling an equal value of the firm's assets

Just after the payment the value of the assets will be decreased by D_{n-1}, so the shareholders will not default as long as this difference remains positive. Here we can implement again this easily on a binomial tree (although not recombining) . Alternatively see Geske(1977) for an analytical solution.

It is always optimal to pay the coupons through asset sales.

Indeed,

- if shareholders pay the coupon out of their pockets, their equity becomes

$$C(V_{t_{n-1}},\ D_{t_n},\ t_n - t_{n-1}) - D_{n-1}$$

- if shareholders decide to sell assets to pay the coupon, their equity becomes

$$C(V_{t_{n-1}} - D_{n-1},\ D_{t_n},\ t_n - t_{n-1})$$

- but the price of a call option (prior to maturity) as a function of the underlying is convex and increasing with first derivative always less than 1,therefore

$$\frac{C(V_{t_{n-1}},\ D_{t_n},\ t_n - t_{n-1}) - C(V_{t_{n-1}} - D_{n-1},\ D_{t_n},\ t_n - t_{n-1})}{D_{n-1}} < 1$$

which implies

$$C(V_{t_{n-1}},\ D_{t_n},\ t_n - t_{n-1}) - D_{n-1} < C(V_{t_{n-1}} - D_{n-1},\ D_{t_n},\ t_n - t_{n-1})$$

- for the pricing part just notice that between coupon dates the assets value follows a GBM but at the coupon dates there is a drop of the asset value by an amount equal to the coupon, IF the coupon is paid by an asset sale. If asset value at the coupon date is below that of the coupon payment, equity is set equal to zero and bondholders take over the firm.

The discrete Jarrow Turnbull Model (1995)

The market

- Trading horizon T
- Trading dates $\{0, 1, ...2, T\}$
- Tradables: Two classes of 0-coupon bonds
  - Default free 0-cpn bonds of all maturities
    $P(t, T)$: the time t value of a default-free 0-cpn, maturing at time T, with face value 1 ($P(t, t) = 1$ for all t)
  - XYZ class of defaultable 0-cpn bonds of all maturities
    $\bar{P}(t, T)$: the time t value of XYZ-class 0-cpn, maturing at time T, with *promised* face value 1
    We talk about the promised face value because the bond is defaultable, i.e. $\bar{P}(t, t)$ may be less than 1.
- There is a money market account that has value $B(t)$ at time t (constructed by investing 1$ at the shortest maturity bond at time zero and rolling it over at all subsequent times)

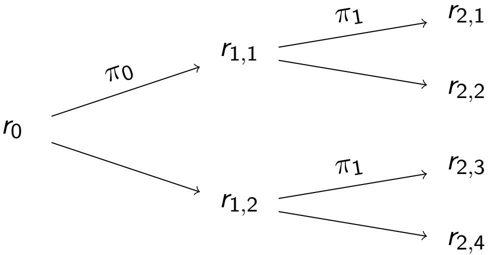
Decomposition of the defaultable bond

- An XYZ firm is thought as having its own currency, let's call it *xyz*.
- Then we can think of a bond issued by XYZ, as a "default-free" bond which always pays at maturity its face value 1*xyz*.
  Denote by $\bar{P}^*(t, T)$ the price of this artificial "default-free" bond when denominated in *xyz*'s
- Then the problem translates into introducing an "exchange rate" between *xyz* and \$
  Define e_t, the amount of \$'s that 1*xyz* is worth at time t.
  Clearly $e_t := \bar{P}(t, t)$ (= the recovery rate at time t)
- Then we can decompose the value of the defaultable bond as

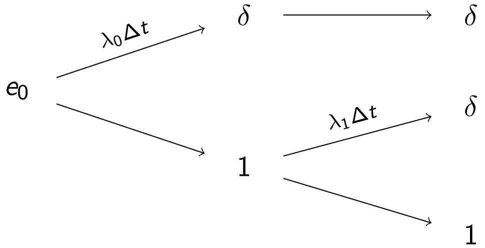
$$\bar{P}(t, T) := \bar{P}^*(t, T) \cdot e_t$$

The two-period model

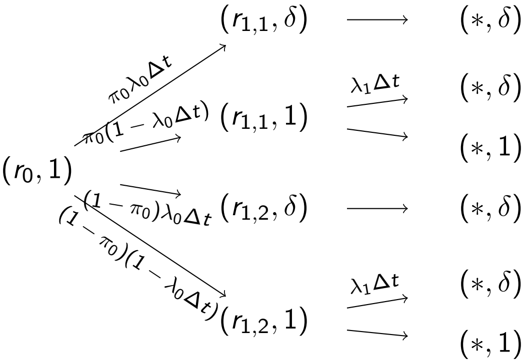
Consider an evolution of the short term risk free inerest rate (the state variable)



Consider also the evolution of the default process ("exchange rate"), where λ_t the intensity of default.



Gathering all the information together and assuming independence of the state variable and the default process we get the following tree representing the evolution of the states of this economy



- For non arbitrage we need the existence of risk-neutral probabilities $\pi_0,\ \lambda_0\Delta t,\ \lambda_1\Delta t$ so that the discounted (by the market account) prices of all tradables are martingales.
- For completeness of the market we need these risk neutral probabilities to be unique.

The default free bond market allows us to determine the risk neutral probability π_0 as follows

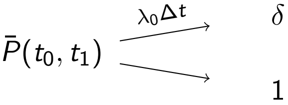
$$\begin{array}{ccccc} t_0 & & t_1 & & t_2 \\ & & P(t_1,t_2)_1 & \longrightarrow & 1 \\ & \overset{\pi_0}{\nearrow} & & & \\ P(t_0,t_2) & & & & \\ & \searrow & & & \\ & & P(t_1,t_2)_2 & \longrightarrow & 1 \end{array}$$

$$P(t_1,t_2)_1 = \exp(-r_{1,1}\Delta t)$$
$$P(t_1,t_2)_2 = \exp(-r_{1,2}\Delta t)$$
$$P(t_0,t_2) = \exp(-r_0\Delta t)\left[\pi_0 P(t_1,t_2)_1 + (1-\pi_0)P(t_1,t_2)_2\right] \Longrightarrow$$

$$\pi_0 = \frac{P(t_0,t_2)\exp(r_0\Delta t) - P(t_1,t_2)_2}{P(t_1,t_2)_1 - P(t_1,t_2)_2}$$

The defaultable bond market will allow us to determine the default probabilities $\lambda_0 \Delta t$ and $\lambda_1 \Delta t$ as follows

- First take the bond maturing at t_1



Then by risk neutrality

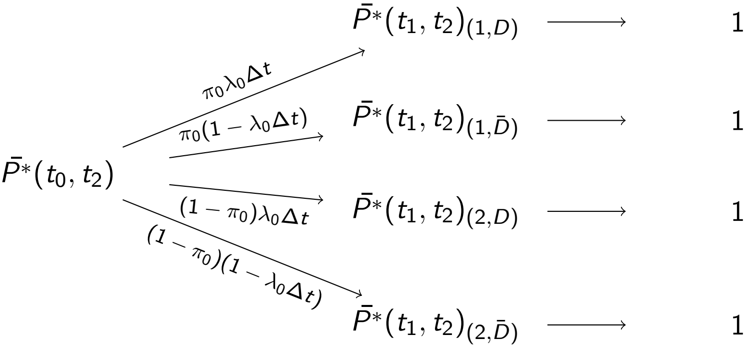
$$\bar{P}(t_0, t_1) = \exp(-r_0 \Delta t) \left[(\lambda_0 \Delta t)\delta + (1 - \lambda_0 \Delta t) \right]$$

which implies that by observing the market price $\bar{P}(t_0, t_1)$ we can infer the risk neutral default probability

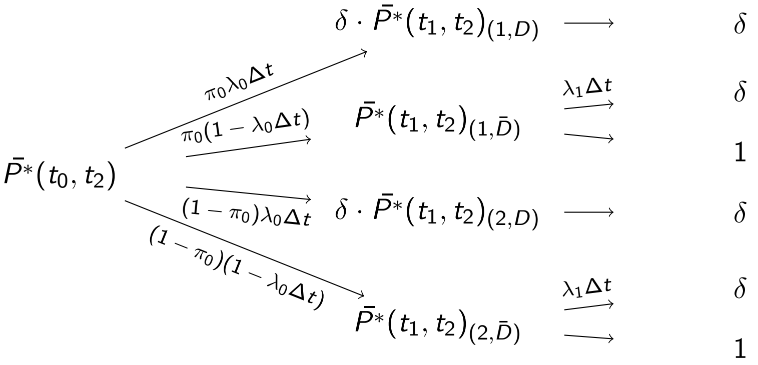
$$\lambda_0 \Delta t = \frac{1 - \bar{P}(t_0, t_1) \exp(r_0 \Delta t)}{1 - \delta}$$

Notice also that we can write $\bar{P}(t_0, t_1) = P(t_0, t_1) E_{\mathcal{Q}}[e_{t_1}]$
i.e. the risky bond price is the discounted price of its expected payoff under the risk neutral measure, with discounting factor the price of the default free bond.

- Now take the bond maturing at t_2. Its evolution in terms of the *xyz* currency is as follows



- Transforming into \$ by using the "exchange rate" e_t we have the following tree



By risk neutrality we have from the interval (t_1, t_2)

$$\bar{P}^*(t_1, t_2)_{(1,D)} = \exp(-r_{1,1}\Delta t) = P(t_1, t_2)_1$$

$$\bar{P}^*(t_1, t_2)_{(2,D)} = \exp(-r_{1,2}\Delta t) = P(t_1, t_2)_2$$

$$\lambda_1 \Delta t = \frac{1 - \bar{P}^*(t_1, t_2)_{(1,\bar{D})} \exp(r_{1,1}\Delta t)}{1 - \delta}$$

$$\lambda_1 \Delta t = \frac{1 - \bar{P}^*(t_1, t_2)_{(2,\bar{D})} \exp(r_{1,2}\Delta t)}{1 - \delta}$$

Furthermore from the time interval (t_0, t_1), we have for the observable $\bar{P}(t_0, t_2)$

$$\bar{P}(t_0, t_2) = \bar{P}^*(t_0, t_2) =$$

$$\pi_0 \lambda_0 \Delta t \delta \bar{P}^*(t_1, t_2)_{(1,D)} + \pi_0 (1 - \lambda_0 \Delta t) \bar{P}^*(t_1, t_2)_{(1,\bar{D})} +$$

$$+ (1 - \pi_0) \lambda_0 \Delta t \delta \bar{P}^*(t_1, t_2)_{(2,D)} + (1 - \pi_0)(1 - \lambda_0 \Delta t) \bar{P}^*(t_1, t_2)_{(2,\bar{D})}$$

so we can solve the system of the last three equations for $\lambda_1 \Delta t$, $\bar{P}^*(t_1, t_2)_{1,\bar{D}}$, $\bar{P}^*(t_1, t_2)_{2,\bar{D}}$.

- Combining all the above it turns out (after a few algebraic manipulations) that

$$\bar{P}(t, T) = P(t, T)E_t(e_t)$$

i.e. the risky bond price is its discounted expected payoff, under the risk neutral probabilities, with discounting factor the default free bond.

- It is quite straightforward, within this framework, to show that a coupon bearing bond is equivalent to a portoflio of zero coupon bonds (each of its coupons is viewed as a zero coupon bond), so the value of such a bond is just the sum of its constituting elements.

Options on risky debt

Within the previous framework the pricing of derivatives written on risky debt is rather simple.
For example consider a European call option written on the two period XYZ bond, with strike K and expiration t_1.
At expiration it has value $C(t_1) = \max[\bar{P}(t_1, t_2) - K, 0]$
Then by risk neutral valuation we have at time t_0 that

$$C(t_0) = \exp(-r_0 \Delta t)(1 - \lambda_0 \Delta t)[\pi_0 C(1)_{(1,\bar{D})} + (1 - \pi_0) C(1)_{(2,\bar{D})}] +$$

$$+ \exp(-r_0 \Delta t)(\lambda_0 \Delta t)[\pi_0 C(1)_{(1,D)} + (1 - \pi_0) C(1)_{(2,D)}]$$

For the hedging of this option we need 3 traded assets and the money market account to replicate its payoff.
We are looking for a portfolio consisting of the two period defaultable 0-cp bond, the one period defaultable 0-cp bond, the two period default free 0-cp bond and the money market account, in such quantities so that at every state of the market at time t_1 this portolio replicates the value of the call option at expiration. Completeness of the market guarantees that this can be done.

Continuous Reduced Form Models

Notation

- $P_{t,T}$: Price at time t of a default-free, 0-cp bond, maturing at time T.
- $\bar{P}_{t,T}$: Price at time t of a defaultable, 0-cp bond, maturing at time T.
- τ: Random time denoting the time of default
- R_τ: Recovery rate at the time of default
- How are the above linked together?
- $y_{t,T}$: The yield of the default-free, 0-cp bond.

$$y_{t,T} := -\frac{\ln P_{t,T}}{T-t} \Leftrightarrow P_{t,T} e^{y_{t,T}(T-t)} = 1$$

- $\bar{y}_{t,T}$: The promised yield of the defaultable, 0-cp bond.

$$\bar{y}_{t,T} := -\frac{\ln \bar{P}_{t,T}}{T-t} \Leftrightarrow \bar{P}_{t,T} e^{\bar{y}_{t,T}(T-t)} = 1$$

Constant risk free interest rate r and zero recovery rate

- $P_{0,T} = e^{-rT}$
- The pay-off of a defaultable bond is

$$1_{\tau>T} = \begin{cases} 1 & \text{if } \tau > T \\ 0 & \text{if } \tau \leq T \end{cases}$$

- Let $\mathcal{Q}_T := \mathcal{Q}(\tau \leq T)$ the risk-neutral probability of default Then $S(T) = \mathcal{Q}(\tau > T) = 1 - \mathcal{Q}_T$ is the probability of surviving up to time T
- Assuming non-arbitrage we have

$$\bar{P}_{0,T} = E_{\mathcal{Q}}[e^{-rT} 1_{\tau>T}] = (1-\mathcal{Q}_T)e^{-rT} \cdot 1 + \mathcal{Q}_T \cdot 0 = (1-\mathcal{Q}_T)P_{0,T}$$

- The previous equation shows that we can infer the risk neutral probability of default from the observed market prices

$$\mathcal{Q}_T = 1 - \frac{\bar{P}_{0,T}}{P_{0,T}}$$

- Using yields the above can be written

$$\mathcal{Q}_T = 1 - e^{\overbrace{(\bar{y}_{0,T} - y_{0,T})}^{-\text{yield spread}} T} = 1 - e^{-s_{0,T} T}$$

Constant risk free rate r and constant recovery rate δ

- The pay-off of a defaultable bond is

$$\left\{ \begin{array}{ll} 1 & \text{if } \tau > T \\ \delta & \text{if } \tau \leq T \end{array} \right\} = 1_{\tau > T} + \delta 1_{\tau \leq T}$$

- Assuming non-arbitrage we have

$$\bar{P}_{0,T} = e^{-rT} E_{\mathcal{Q}}[1_{\tau > T} + \delta 1_{\tau \leq T}]$$
$$= e^{-rT} \left[(1 - \mathcal{Q}_T) \cdot 1 + \mathcal{Q}_T \cdot \delta\right] = (1 - \mathcal{Q}_T(1 - \delta))P_{0,T}$$

- This implies for the risk neutral PD

$$\mathcal{Q}_T = \frac{1}{1-\delta} - \frac{\bar{P}_{0,T}}{P_{0,T}(1-\delta)}$$

- or in terms of the yield spread $s_{0,T} = \bar{y}_{0,T} - y_{0,T}$

$$\mathcal{Q}_T = \frac{1 - e^{-s_{0,T}}}{1-\delta}$$

Deterministic risk free interest rate r_t and deterministic recovery rate R_t

- The pay-off of a defaultable bond is

$$\begin{cases} 1 & \text{if } \tau > T \\ R_\tau & \text{if } \tau \leq T \end{cases} = 1_{\tau > T} + R_\tau 1_{\tau \leq T}$$

- The discounted pay off of the defaultable bond is

$$e^{-\int_0^T r(u)du} 1_{\tau > T} + e^{-\int_0^\tau r(u)du} R_\tau 1_{\tau \leq T}$$

- Assuming non-arbitrage we have

$$\bar{P}_{0,T} = E_{\mathcal{Q}}[e^{-\int_0^T r(u)du} 1_{\tau > T} + e^{-\int_0^\tau r(u)du} R_\tau 1_{\tau \leq T}]$$

Modeling the default time τ

- Define the hazard rate or intensity of default $\lambda(t)$ by

$$\lambda(t)dt = \mathcal{Q}(t < \tau \leq t + dt | \tau > t)$$

- This is the probability of default in the small time interval $(t, t + dt]$ provided that default has not occured up to time t

Proposition

- $\mathcal{Q}(\tau > t) = e^{-\int_0^t \lambda(u)du}$
- *The cdf of τ is given by*

$$F_\tau(t) = \mathcal{Q}(\tau \leq t) = 1 - e^{-\int_0^t \lambda(u)du}$$

- *The pdf of τ is given by*

$$f_\tau(t) = \lambda(t)e^{-\int_0^t \lambda(u)du}$$

Corollary

The price of a defaultable 0-cp bond with zero recovery rate is given as

$$\bar{P}_{0,T} = e^{-\int_0^T r(u)du}\mathcal{Q}(\tau > T) = e^{-\int_0^T (r(u)+\lambda(u))du}$$

Proof.

Denote by $S(t) = \mathcal{Q}(\tau > t)$ the probability of surviving up to time t. Then

$$S(t+dt) = \mathcal{Q}(\tau > t+dt) = \mathcal{Q}(\tau > t+dt|\tau > t)S(t) =$$

$$[1-\mathcal{Q}(\tau \leq t+dt|\tau > t)]S(t) = [1-\mathcal{Q}(t < \tau \leq t+dt|\tau > t)]S(t) \Rightarrow$$

$$S(t+dt) = (1-\lambda(t)dt)S(t)$$

$$\frac{1}{S(t)}dS(t) = -\lambda(t)dt \Rightarrow S(t) = S(0)e^{-\int_0^t \lambda(u)du}$$

and since $S(0) = 1$ we have

$$\mathcal{Q}(\tau > t) = e^{-\int_0^t \lambda(u)du}$$

The rest of the proposition follows directly □

back to the deterministic recovery rate R_t

Proposition

If the risk free rate r_t and the recovery rate R_t are deterministic functions of time, then the current price of a defaultable 0-cp bond, maturing at T and with default intensity rate $\lambda(t)$ is given by:

$$\bar{P}_0^T = e^{-\int_0^T (r(u)+\lambda(u))du} + \int_0^T \lambda(t) R_t e^{-\int_0^t (r(u)+\lambda(u))du} dt$$

Proof.

As we have seen

$$\bar{P}_{0,T} = E_{\mathcal{Q}}[e^{-\int_0^T r(u)du} 1_{\tau > T} + e^{-\int_0^\tau r(u)du} R_\tau 1_{\tau \leq T}]$$

By the previous proposition we get for the second term of the above sum that

$$E_{\mathcal{Q}}\left[e^{-\int_0^\tau r(u)du} R_\tau 1_{\tau \leq T}\right] = \int_0^T \lambda(t) R_t e^{-\int_0^t (r(u)+\lambda(u))du} dt$$

and the result follows directly □

$\bar{P}_{s,T}$

The price of the defaultable bond at a future day $s \leq T$ depends on whether the default has alrady taken place or not. In this case

$$\bar{P}_s^T = 1_{\tau > s}\left[e^{-\int_s^T (r(u)+\lambda(u))du} + \int_s^T \lambda(t) R_t e^{-\int_s^t (r(u)+\lambda(u))du} dt\right]$$

Stochastic risk free interest rate, stochastic recovery, stochastic intesity of default

In brief it goes like this. Let $(r_t)_t$ and $(R_t)_t$ be risk free interest rate and the recovery rate stochastic processes, adapted to some filtration $(\mathcal{F}_t)_t$. Consider the σ algebras $\mathcal{H}_t = \sigma(1_{\tau \leq s} : s \leq t)$ and then the σ-algebra $\mathcal{G}_t = \sigma(\mathcal{H}_t \cup \mathcal{F}_t)$ so that we incorporate information about default. Then the price of the defaultable bond at time t is

$$\bar{P}_{t,T} = E_{\mathcal{Q}}\left[e^{-\int_t^T r_u du} 1_{\tau > T} + e^{-\int_t^\tau r_u du} R_\tau 1_{t < \tau \leq T} | \mathcal{G}_t\right]$$

In the same spirit as before one defines the default intensity

$$\lambda_t dt = \mathcal{Q}(t < \tau < t + dt | \{\tau > t\}, \mathcal{F}_t)$$

This is a stocahstic process adapted to $\mathcal{F}_t$ Then it can be proved that::

$$\bar{P}_{t,T} = 1_{\tau > t} \left[E_{\mathcal{Q}} \left(e^{-\int_t^T (r_u + \lambda_u) du} | \mathcal{F}_t \right) + E_{\mathcal{Q}} \left(\int_t^T \lambda_s \mathbb{R}_s e^{-\int_t^w (r_u + \lambda_u) du} | \mathcal{F}_t \right) \right.$$

(see book of Bielecki and Rutkowski, Springer, 2002)

Portfolio Considerations - The Vasicek Model

The Vasicek model

- Portfolio of N firms (obligors)
- Interested in the distribution of
  - (i) the default rate (i.e. percentage of defaulted credits)
  - (ii) the portfolio credit loss
- Denote by

$(V_{n,t})$	The Assets value process of the n-th firm
$(B_{n,t})$	The liabilities process of the n-th firm. Only 0-coupon bond with face value D_n, maturing at T.
$(E_{n,t})$	Equity process of the n-th firm

- The n-th firm defaults if $V_{n,T} < D_n$

Let

- $L_n := 1_{\{V_{n,T} < D_n\}}$, the default indicator variable of the n-th firm
- $L := \sum_{n=1}^{N} L_n$, the total number of defaults in the portfolio.
- $\Omega := L/N$, the portfolio default rate
- $F(\omega, \;\; p, \;\; \rho_T) := \mathcal{P}(\Omega \leq \omega)$, the cumulative distribution function of the portfolio default rate.

- Assume assets value dynamics

$$dV_{n,t} = V_{n,t}\mu_n dt + \sigma_n V_{n,t} dW_{n,t}$$

which implies

$$\ln V_{n,T} = \ln V_{n,0} + \left(\mu_n - \frac{\sigma_n^2}{2}\right) T + \sigma_n \sqrt{T} \underbrace{X_{n,T}}_{N(0,1)}$$

- Then the probability of default of the n-th firm is given by

$$p_n := \mathcal{P}(V_{n,T} < D_n) = \mathcal{P}(X_n < k_n) = \Phi(k_n)$$

where Φ the cumulative standard normal and k_n is the default barrier

$$k_n = \frac{\ln\left(\frac{D_n \exp(-\mu_n T)}{V_{n,0}}\right)}{\sigma_n \sqrt{T}} + \frac{\sigma_n \sqrt{T}}{2}$$

- Therefore a 1-1 correspondence is established between PDs and default barriers via the relation $p_n = \Phi(k_n)$

- The way of deriving the PDs is not important in this model. In fact we could have been given the individual PDs exogenously or through some other model and then compute the corresponding default barriers k_n via $k_n = \Phi^{-1}(PD)$
- However PDs are not enough for our purpose. We need a correlation structure of defaults.
- Defaults are endogenous to Merton's model, driven by assets dynamics
- The only random variables involved are the standard normals $X_{n,T}$.
- Define

$$\rho_{n,m,T} = correl(X_{n,T}, X_{m,T})$$

Assumption 1: Common correlation coefficient

- Assume a common correlation among the random variables X_n

$$\rho_{n,m,T} = \rho_T \text{ for any} n \neq m$$

- We think of the uncertainty about each firms assets as relying on two independent risk factors
  - A systematic risk factor that affects all firms in the same way
  - An idiosyncratic factor, one for each firm, independent accross firms and independent w.r.t the systematic factor

- In particular we can write

$$X_{n,T} = \sqrt{\rho_T} Y_T + \sqrt{1-\rho_T} \epsilon_{n,T}$$

with $Y_T, \epsilon_{1,T}, \cdots, \epsilon_{N,T}$ i.i.d. standard normal.
- Y_T is the systematic factor and $\epsilon_{n,T}$ are the idiosyncratic factors
- It is not necessary to assume normality for the Y_T and $\epsilon_{n,T}$. In fact we can assume that Y_T has a distribution Φ_Y and $\epsilon_{n,T}$ have distribution Φ_ϵ

conditioning on Y_T

- Clearly conditioning upon Y_T the $X_{n,T}$s are independent and therefore the default variables $1_{X_{n,T}<k_n}|_{Y_T}$ are also independent
- The idea is to condition upon the systematic factor, which will result in independent default probabilities and then integrate accross all the values of the systematic factor
- Define $p_n(Y_T) = \mathcal{P}(X_{n,T} < k_{n,T}|Y_T)$ the probability of default of the n-th firm conditional on the value of the systematic risk factor.
- Remark that $p_n(Y_T) = E(1_{\{X_{n,T}<k_n\}}|Y_T)$

- Then

$$
\begin{aligned}
p_n(Y_T) &= \mathcal{P}\left(\sqrt{\rho_T}Y_T + \sqrt{1-\rho_T}\epsilon_{n,T} < k_{n,T} | Y_T\right) \\
&= \mathcal{P}\left(\epsilon_{n,T} < \frac{k_{n,T} - \sqrt{\rho_T}Y_T}{\sqrt{1-\rho_T}} | Y_T\right) \\
&= \Phi_\epsilon\left(\frac{k_{n,T} - \sqrt{\rho_T}Y_T}{\sqrt{1-\rho_T}}\right) \\
&= \Phi_\epsilon\left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho_T}Y_T}{\sqrt{1-\rho_T}}\right)
\end{aligned}
$$

- Let $L_n := 1_{\{V_{n,T}<D_n\}}$, the default indicator variable of the n-th firm
- $L := \sum_{n=1}^{N} L_n$, the total number of defaults in the portfolio.
- $\Omega := L/N$, the portfolio default rate
- $F(\omega, \ p, \ \rho_T) := \mathcal{P}(\Omega \leq \omega)$, the cumulative distribution function of the portfolio default rate.

Assumption 2: Common probability of default

- Assume all firms have the same probability of default

$$p := p_1 = \cdots = p_N$$

- Then all firms have the same conditional (upon Y_T) probability of default:

$$p(Y_T) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho_T}\,Y_T}{\sqrt{1-\rho_T}}\right)$$

- $L_n := 1_{\{V_{n,T} < D_n\}}$, the default indicator variable of the n-th firm
- $L := \sum_{n=1}^{N} L_n$, the total number of defaults in the portfolio.
- $\Omega := L/N$, the portfolio default rate
- $F(\omega,\ \ p,\ \ \rho_T) := \mathcal{P}(\Omega \leq \omega)$, the cumulative distribution function of the portfolio default rate.

The distribution of the number of defaults

$$\mathcal{P}(L=n) = E(1_{L=n}) = E(E(1_{L=n}|Y_T)) = \int_{-\infty}^{+\infty} E(1_{L=n}|Y_T=y)f_Y(y)dy$$

$$\mathcal{P}(L=n) = \int_{-\infty}^{+\infty} \mathcal{P}(L=n|Y_T=y)f_Y(y)dy$$

Having assumed a common probability of default and bearing in mind the independence of the default probabilities when conditioned upon Y_T, we have that $(L=n|Y_T=y)$ follows binomial distributiona and thus

$$\mathcal{P}(L=n|Y_T=y) = \binom{N}{n} p(y)^n(1-p(y))^{N-n}$$

Combining these last two relations with $p(y) = \Phi_\epsilon\left(\frac{\Phi^{-1}(p_n)-\sqrt{\rho_T}y}{\sqrt{1-\rho_T}}\right)$ we obtain

- $$\mathcal{P}(L = n) =$$
$$\int_{-\infty}^{+\infty} \binom{N}{n} \left[\Phi_\epsilon \left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho_T} y}{\sqrt{1-\rho_T}} \right) \right]^n$$
$$\left[1 - \Phi_\epsilon \left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho_T} y}{\sqrt{1-\rho_T}} \right) \right]^{N-n} f_Y(y) dy$$

 and

- $$\mathcal{P}(L \leq m) =$$
$$\sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{+\infty} \left[\Phi_\epsilon \left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho_T} y}{\sqrt{1-\rho_T}} \right) \right]^n$$
$$\left[1 - \Phi_\epsilon \left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho_T} y}{\sqrt{1-\rho_T}} \right) \right]^{N-n} f_Y(y) dy$$

Assumption 3: The portfolio is large ($N \to \infty$)

When the portfolio is large things get even better since independence of conditional (upon Y) defaults allows the use of the law of large numbers .
Indeed the default rate (i.e. the percentage of defaults) is $\Omega = L/N = (\sum_{i=1}^{N} L_i)/N$
Conditional on Y this is the average of the sum of N iid random variables and thus it converges to the common mean $p(y)$, i.e.

$$\mathcal{P}(\Omega = p(y)|Y = y) = 1$$

In other words we can say that Ω converges to $p(Y)$ as N gets large

Then

$$\mathcal{P}(\Omega \leq \omega) = \mathcal{P}(p(Y) \leq \omega)$$

$$= \mathcal{P}\left(\Phi_\epsilon\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \leq \omega\right)$$

$$= \mathcal{P}\left(Y \geq \frac{\Phi^{-1}(p) - \sqrt{1-\rho}\Phi_\epsilon^{-1}(\omega)}{\sqrt{\rho}}\right)$$

$$= 1 - \Phi_Y\left(\frac{\Phi^{-1}(p) - \sqrt{1-\rho}\Phi_\epsilon^{-1}(\omega)}{\sqrt{\rho}}\right)$$

When $\rho = 0$ then defaults are independent, so $\Omega = p$ with probability 1

When $\rho = 1$ then there is perfect correlation and $\Omega = 0$ with probability $1 - p$ and $\Omega = 1$ with probability p

In case that $\Phi_Y = \Phi_\epsilon = \Phi$ is standard normal, the previous result gets the form

$$\mathcal{P}(\Omega \leq \omega) = \Phi\left(-\frac{\Phi^{-1}(p) + \sqrt{1-\rho}\Phi^{-1}(\omega)}{\sqrt{\rho}}\right)$$

Furthermore, differentiating with respect to ω we obtain the density function of the default rate:

$$f_\Omega(\omega) = sqrt\frac{1-\rho}{\rho}\exp\left[\frac{1}{2}\left(\Phi^{-1}(\omega)\right) - \frac{1}{2\rho}\left(\Phi^{-1}(p) - \sqrt{1-\rho}\Phi^{-1}(\omega)\right)^2\right]$$

Moreover it can be shown that

$$E(\Omega) = p$$

and

$$Var(\Omega) = \Phi_2(\Phi^{-1}(p), \Phi^{-1}(p))$$

Assuming further that all firms have the same recovery rate RR and that the size of each loan is similar then the percentage loss of the total initial value of the portfolio can be given as $(1 - RR)\Omega$

Credit Derivatives

Credit Derivatives

- Financial instruments that isolate the credit risk of some underlying defaultable instruments (e.g. loans, bonds).
- This is achieved by linking the cashflows of the credit derivatives to the default losses of the underlying reference entities.
- They allow investors to trade credit risk and thus modify their credit exposure according to their beliefs.
- Typically they look like bilateral insurance contracts with the buyer buying protection against losses (due to default of some reference entity) and the seller selling this protection.
    - Buy protection=short credit risk
    - sell protection = long credit risk

Building blocks

- *Reference Entity:* The entity that we trade its credit risk (single entity or portfolio)
- *Notional Amount:* The amount of credit that is being traded
- *Term (or maturity):* Refers to the expiry of the credit protection
- *Premium (or spread):* The periodic fee that is being paid by the buyer to the seller. It is expressed as annualized percentage of the notional amount and is usually paid quarterly.
- *Credit events (defaults):* Events that "trigger" the contract's protection process (e.g. bankruptcy, missing of payment, downgrading etc)
- *Settlement:* The procedure by which the seller compensates the buyer for the loss caused by the credit event

Credit Default Swaps (CDS)

Mechanics

- Single name credit derivative
- *Protection Buyer* ⟵ – – – – – – – – default loss compensation – – – – – – – – *Protection seller*

  Protection Buyer – – – – – – – – premium – – – – – – – – ⟶ *Protection seller*
- Premium payments stop being paid when the credit event takes place (otherwise at maturity).
- When the credit event takes place between two payment dates, the protection buyer pays the accrued premium.
- *Premium (or fixed) leg:* Represents the payments from the CDS buyer to the CDS seller
  Default (or protection) leg: Represents the payment from the seller to the buyer

Settlement in the case of default

- Physical settlement

Protection Buyer ← Notional — — — *Protection seller*

Protection Buyer — — — Underlying bond or loan → *Protection seller*

- Cash settlement

Protection Buyer ← Notional (1–RR) — — — *Protection seller*

Example

Example

Reference Entity:	XYZ	
Notional Amount:	10 million $	Suppose that
Term:	3 years	
Annual Premium:	300bps (=3%) paid quarterly	

XYZ defaults 7 months after the origination of the CDS and that the recovery rate is 40%.

Then the cashflows to the buyer of the CDS are:

Time	Outflows	Inflows
3m	75.000$ $(=\frac{3\%}{4}10mln)$	0$
6m	75.000$ $(=\frac{3\%}{4}10mln)$	0$
7m	25.000$ $(=\frac{3\%}{12}10mln)$	6mln$ $(=10mln(1-40\%))$

Valuation issues

- The value of a CDS is the expected (under the risk neutral measure) present value of its cashflows
- At origination, the spread (premium) is chosen so that the CDS has zero value (i.e. no initial exchange of money)
  This means that the spread is chosen so that:
  Expected discounted value of all premium payments = expected discounted value of default loss
- The accrued premium, when default does not happen exactly on payment dates, has to be taken into account

One period toy example

Assumptions

- The premium s is paid at the end of the period
- Default can only happen at the end of the period with risk neutral probability of default p
- Recovery rate is known and equal to R
- Notional N
- Risk free interest rate r

One period toy example

Then

- $EPV(\text{premium leg}) = \frac{N(1-p)s}{1+r}$
- $EPV(\text{protection leg}) = \frac{Np(1-R)}{1+r}$
- Equating the two legs implies

$$s = \frac{p(1-R)}{1-p}$$

- Alternatively if we observe the CDS spread s we can infer the probability of default

$$p = \frac{s}{s+1-R}$$

Valuation

Assumptions

- Complete market and absence of arbitrage
- indepenence of default process and risk free interest rates process under risk neutral measure
- The buyer and seller of the CDS do not default during the life of the CDS
- Constant and exogenous recovery rate R
- The CDS is originated at $t_0 = 0$ and there are n contractual payment dates $t_1, \ldots, t_n$ with t_n the maturity of the CDS. Let $\Delta t_i = t_i - t_{i-1}$ in years.
- Notional N, Annual spread s, annual premium $\varpi = sN$
- $S(t)$: survival probability of the reference entity up to time t
- $D(t)$: risk-free discount factor for the time period $[0, t]$

Valuation of the premium leg

- The value of the premium leg is the expected present value of all the possible payments. This includes payments at the payment dates and accrual payments between payment dates.
- Expected present value of payment at $t_1, \ldots, t_n$
  - $sN\Delta t_i =$ premium due at t_i, if survived up to time t_i
  - $sN\Delta t_i D(t_i) =$ its present value
  - $sN\Delta t_i D(t_i) S(t_i) =$ its expected present value
  - Summing up

$$sN\sum_{i=1}^{n}\Delta t_i D(t_i) S(t_i)$$

- But we have not taken into account the possibility of accrual payments

- Expected present value of accrual payments.
- Consider premium accrual period corresponding to $t \in (t_{i-1}, t_i)$
- $sN(t - t_{i-1})/360 =$ premium due at t, if survived up to time t and then defaulted at the next infinitessimal interval dt
- The probability of having survived up to t and then defaulting within the next small interval dt is $S(t) - S(t + dt) = d(1 - S(t))\,(= S(t)\lambda(t)dt$ when Poisson)
- Then the expected present value of the possible accrual payments of the period (t_{i-1}, t_i) is $sN \int_{t_{i-1}}^{t_i} \frac{(t - t_{i-1})}{360} D(t) d(1 - S(t))$
- Summing across all time intervals $sN \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t - t_{i-1})}{360} D(t) d(1 - S(t))$

Therefore

$$EPV(\text{premium leg}) =$$

$$sN\left[\sum_{i=1}^{n}\Delta t_i D(t_i)S(t_i) + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t-t_{i-1})}{360}D(t)d(1-S(t))\right]$$

Valuation of the default leg and breakeven spread

- The probability of having survived up to t and then defaulting within the next small interval dt is $S(t) - S(t + dt) = d(1 - S(t))\,(= S(t)\lambda(t)dt$ when Poisson)
- Therefore

$$EPV(\text{default leg}) = (1 - R)N \int_0^{t_n} D(t)d(1 - S(t))$$

- Equating the premium leg to the default leg we get the breakeven spread

$$s = \frac{(1 - R)\int_0^{t_n} D(t)d(1 - S(t))}{\sum_{i=1}^n \Delta t_i D(t_i)S(t_i) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(t-t_{i-1})}{360} D(t)d(1 - S(t))}$$

Valuation using Poisson process setup

- Default arrives unexpectedly in Poisson process with arrival rate λ_t
  $\mathcal{P}(\tau \in [t, t + dt]) = \lambda_t dt$
  $\mathcal{P}(\text{survival up to } t) = S(t) = \exp(-\int_0^t \lambda_s ds)$
  $dS(t) = -\lambda_t S(t) dt$
  $d(1 - S(t)) = -\lambda_t S(t) dt$
- Then the breakeven spread of the CDS is given

$$s = \frac{(1 - R)\int_0^{t_n} D(t)S(t)\lambda(t)dt}{\sum_{i=1}^{n} \Delta t_i D(t_i)S(t_i) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t - t_{i-1})}{360} D(t)S(t)\lambda(t)dt}$$

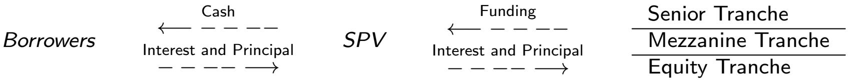
Collateralized Debt Obligations (CDO)

Mechanics

- Multiname credit derivative
- A CDO is a SPV with assets, liabilities and a manager
- The assets of the CDO typically consist of a diversified portfolio of defaultable instruments (loans, bonds etc)
- This portfolio has credit risk. This credit risk is "sliced" into priority tranches and these tranches are sold to investors.
- The investors will bear the potential losses of the portfolio in exchange for a periodic fee.
- Cash CDO: the underlying portfolio consists of the actual loans, bonds etc.

  Synthetic CDO: the underlying portfolio consists of CDSs (a portfolio of CDS's achieves the same credit exposure

A typical CDO structure



The income from the borrowers (assets) is paid to the investors according to their seniority. First senior notes are paid, then the mezzanine notes and finally whatever remains is paid to the equity notes. In this sense the senior notes are much safer than the reference portfolio of assets, since they will start facing losses only if the lower tranches cannot absorb them.

- The CDO tranches are rated by rating agencies according to their ability to service their debt through the cashflows generated by the underlying portfolio of assets.
- The tranches are defined by a lower and an upper attachment point. The holder of a tranche with lower attachment point A_L and upper attachment point A_U will suffer all losses in excess of A_L and up to A_U, as percentages of the iinitial value of the reference portolio.

CDO capital structure

Classes	Rating	Premium	% of capital structure	Attachment points
Supersenior	AAA	LIBOR+0,5%	70%	30%-100%
Senior	A	LIBOR+1,5%	14%	16%-30%
Mezzanine 1	BBB	LIBOR+2,5%	8%	8%-16%
Mezzanine 2	C	LIBOR+6,5%	4%	4%-8%
Equity	Non rated	27%	4%	0%-4%

- T: maturity of the CDO
- M: initial value of the portfolio (at time t_0)
- Z_t: percentage losses in the portfolio at time t
- At time t the total loss corresponding to tranche j is $Z_{j,t}M$ where:

$$Z_{j,t} = \min(Z_t, K_{U_j}) - \min(Z_t, K_{L_j})$$

- Losses are paid by tranche holders at the payment dates $t_1, ..., t_n = T$, usually every quarter. Let $h = t_{i+1} - t_i$ the distance between payment dates, expressed in years.
- Then the loss suffered from payment date t to payment date $t + h$ is $(Z_{j,t+h} - Z_{j,t})M$

- $\Gamma_{j,t}$: the outstanding pricipal of tranche j at time t. THen

$$\Gamma_{j,t} = \underbrace{(K_{U_j} - K_{L_j})M}_{\text{initial notional}} - \underbrace{Z_{j,t}M}_{\text{loss up to } t}$$

$$= \begin{cases} (K_{U_j} - K_{L_j})M, & \text{if } Z_t < K_{L_j} \\ (K_{U_j} - Z_t)M, & \text{if } K_{L_j} \leq Z_t \leq K_{U_j} \\ 0, & \text{if } Z_t > K_{U_j} \end{cases}$$

- Let s_j the (annualised) percentage premium on the outstanding notional of tranche j that tranche j holders receive on payment dates.
- At each payment date $t = h, 2h, ..., nh = T$, tranche j receives

$$s_j h \Gamma_{j,t}$$

and pays

$$(Z_{j,t} - Z_{j,t-h})M$$

- The outstanding capital $\Gamma_{j,t}$ is decreasing function of the total portfolio loss $Z_t M$ and becomes zero as soon as $Z_t \geq K_{U_j}$. In such a case tranche j has lost all of its capital and there are no further payments to be made or received.

Pricing

- What is the appropriate premium s_j for each tranche j?
- Choose it so that:Expected present value of paid cashflows = Expected present value of received cashflows
- Assume absence of arbitrage, complete markets, independence of risk free rates and default process

- Received cashflows leg at t_0

$$X_{R,j} = \sum_{k=1} nD(t_k)s_j hE[(K_{U_j} - K_{L_j})M - Z_{j,t_k}M]$$

with $D(t_k)$ the discount factor for the time period from t_0 to t_k

- Paid cashflows leg at t_0

$$X_{P,j} = \sum_{k=1} nD(t_k)E[(Z_{j,t_k} - Z_{j,t_{k-1}})M]$$

- Choose s_j so that $X_{R,j} = X_{P,j}$ which implies

$$s_j = \frac{\sum_{k=1} nD(t_k)(E[Z_{j,t_k}] - E[Z_{j,t_{k-1}}])}{\sum_{k=1} nD(t_k)h(K_{U_j} - K_{L_j} - E[Z_{j,t_k}])}$$

- Need to calculate $E[Z_{j,t_k}] = E[\min(Z_{t_k}, K_{U_j}) - \min(Z_{t_k}, K_{L_j})]$ for $k = 1, ..., n$ (since $E[Z_{j,t_0}] = 0$)
- Need distribution functions of Z_{t_k} for all $k = 1, ..., n$

- Let's recall Vasicek's model for a large homogeneous portfolio
- Assumptions
  - Default correlations:(ρ_t) equal across all portfolio credits (due to a common systematic factor)
  - Individual probabilities of default: (p_t) known and equal accross all portfolio credits
  - Number of firms in portfolio: large ($N \to \infty$)
  - Recovery rate: (RR_t) deterministic and common to all portfolio credits
  - Exposures: similar accross all firms
- Then
  - The time t portfolio default rate (Ω_t) has distribution

$$F(\omega; p_t; \rho_t) = \mathcal{P}(\Omega_t \leq \omega) = \Phi\left(-\frac{\Phi^{-1}(p_t) + \sqrt{1-\rho_t}\Phi^{-1}(\omega)}{\sqrt{\rho_t}}\right)$$

  - The time t percentage loss Z_t of the total initial value of the portfolio is

$$Z_t = (1 - RR_t)\Omega_t$$

Then the expected loss of tranche j at time t_K is

$$E[Z_{j,t_k}] =$$

$$\int_0^1 \left(\min\{(1-RR_{t_k})\omega,\ K_{U_j}\} - \min\{(1-RR_{t_k})\omega,\ K_{L_j}\}\right) dF(\omega; p_t; \rho_t)$$

The integral can be evaluated numerically and then the spread s_j can be calculated.

- It is usual practice to
  - Assume constant recovery rate across time
  - Assume constant correlation across time
  - Estimate correlation from correlation of equity returns
  - Estimate probabilities of default as those implied by a reduced form model calibrated with bond or CDS prices

The following list of references and suggested readings consists of books, articles and notes that this presentation has been based on, together with resources that are considered to be of interest for reference or further study. The articles listed here are restricted to the very basic ones since the review articles mentioned below offer a rather complete guideline to the literature.

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